STUDENT NAME:

Linear Algebra Graduate Comprehensive Exam, May 2017
Worcester Polytechnic Institute

Work out 6 of the following problems
Write down detailed proofs of every statement you make.
No Books. No Notes. No calculators.

1. Let $\mathbb{R}^{n\times n}$ be the set of n by n real matrices, equipped by the metric given by any norm.
(i). Show that the subset of diagonalizable matrices of $\mathbb{R}^{n\times n}$ is dense.
(ii). Is the set of diagonalizable matrices of $\mathbb{R}^{n\times n}$ open? (Prove or disprove).

2. Let A and B be Hermitian $n \times n$ complex matrices.
(i). If A is positive definite, show that there exists an invertible matrix P such that $P^*AP = I$ and P^*BP is diagonal.
(ii). If A is positive definite and B is positive semidefinite, show that
$$\det(A + B) \geq \det(A).$$

3. Let $V = \mathbb{R}^5$ and T is a linear transformation from V to itself defined by $T(a, b, c, d, e) = (2a, 2b, 2c + d, a + 2d, b + 2e)$.
(i). Find the characteristic polynomial and minimal polynomial of T.
(ii). Determine a basis of $\mathbb{R}^5$ consisting of eigenvectors and generalized eigenvectors of T.
(iii). Find the Jordan form of T with respect to your basis.

4. Fredholm Alternative: Let A be an $m \times n$ real matrix and $b \in \mathbb{R}^m$. Show that exactly one of the following systems has a solution:
(i). $Ax = b$
(ii). $A^T y = 0, \quad y^T b \neq 0$.

5. Let U and W be subspaces of the finite-dimensional inner product space V.
(i). Prove that $U^\perp \cap W^\perp = (U + W)^\perp$.
(ii). Prove that
$$\dim(W) - \dim(U \cap W) = \dim(U^\perp) - \dim(U^\perp \cap W^\perp).$$

6. Find a basis for the intersection of the subspace of $\mathbb{R}^4$ spanned by $(1, 1, 0, 0)$, $(0, 1, 1, 0)$, $(0, 0, 1, 1)$ and the subspace spanned by $(1, 0, t, 0)$, $(0, 1, 0, t)$, where t is given.

7. Let A and B be $n \times n$ complex matrices such that $AB = BA$. Show that if A has n distinct eigenvalues, then A, B and AB are all diagonalizable.